

Geometric and diffractive orbits in the scattering from confocal hyperbolas

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We study the scattering resonances between two confocal hyperbolas and show that the spectrum is dominated by the effect of a single periodic orbit. There are two distinct cases, depending on whether the orbit is geometric or diffractive. A generalization of periodic orbit theory allows us to incorporate the second possibility. In both cases, we also perform a Wentzel-Kramers-Brillouin (WKB) analysis. Although it is found that the semiclassical approximations work best for resonances with large energies and narrow widths, there is reasonable agreement even for resonances with large widths — unlike the two disk scatterer. We also find agreement with the next order correction to periodic orbit theory.

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In recent years there has been growing interest in understanding the extent to which knowledge of classical mechanics can be used to understand quantum systems [1]. For sufficiently hyperbolic systems, periodic orbits provide an efficient means of calculating spectra semiclassically. One class of problem which has been fruitfully studied is the n -disk scatterer in two dimensions [2, 3]. It has been shown that in some situations periodic orbit theory gives the energies and widths of the scattering resonances to great accuracy — typically several decimal places.

However, this accuracy is usually only for the leading family of resonances. There exist other resonances with larger widths deeper in the complex momentum plane. These are badly approximated by periodic orbit theory because it fails to consider classical paths which, although not trajectories, satisfy the stationary phase condition. Recent work [4, 5] has shown that inclusion of diffractive paths — so-called creeping orbits — recovers the qualitative features of the exact spectrum, including the lower order families. However, the quantitative agreement is not as good as for the leading family of resonances which are unaffected by creeping.

It is useful to consider a system which has no creeping so as to study the full spectrum of resonances. One such system is a pair of confocal hyperbolas. Like the two-disk scatterer there is only one periodic orbit, which is unstable. However, unlike the two disk scatterer, there are no creeping orbits. Since the system is separable we can apply WKB techniques as well as periodic orbit theory but this is more difficult and is not as intuitive. In addition, we can evaluate the resonances for the special case where one or both hyperbolas are close to half planes. Then we must consider edge diffraction and this is included in the periodic orbit analysis.

We want to solve the Schrödinger equation in a domain between two confocal hyperbolas, as shown for example in Figs. 1 and 2. This is simply the Helmholtz equation $(\nabla^2 + k^2)\Psi = 0$ with the boundary conditions that on the two hyperbolas Ψ vanishes and that in the re-

gion between them Ψ approaches $f(\phi) \exp(ikr)/\sqrt{r}$ for large r (where r and ϕ are polar coordinates). We will work in hyperbolic-elliptic coordinates μ and θ , defined by $x = a \cosh \mu \cos \theta$ and $y = a \sinh \mu \sin \theta$ [6]. Curves of constant μ are ellipses and curves of constant θ are hyperbolas. In both cases the foci are at $y = 0$ and $x = \pm a$. We take the coordinates to be in the ranges $0 \leq \theta \leq \pi$ and $-\infty < \mu < \infty$ and label the right and left hyperbolas, respectively, by θ_1 and θ_2 so that $\theta_2 > \theta_1$. We define length units such that the closest distance between the

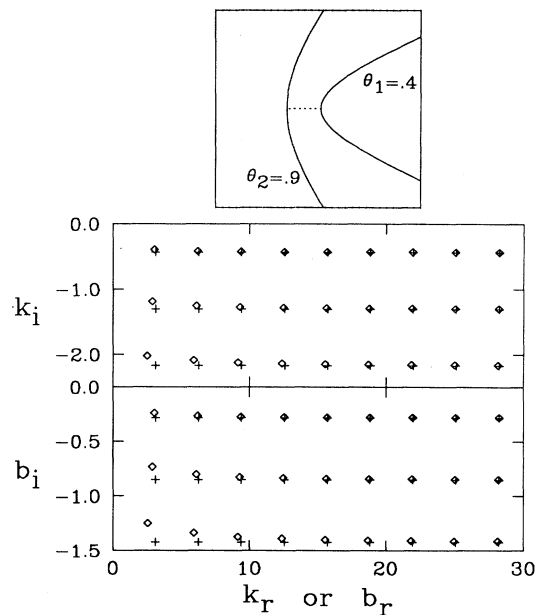


FIG. 1. The top box is the configuration space for $\theta_1 = 0.4$ and $\theta_2 = 0.9$. The sole periodic orbit is shown as a dotted line. The middle and bottom boxes show the spectra of k and b , respectively, plotted in the complex plane. The exact quantum resonances are represented as open diamonds and the semiclassical predictions are represented as crosses.

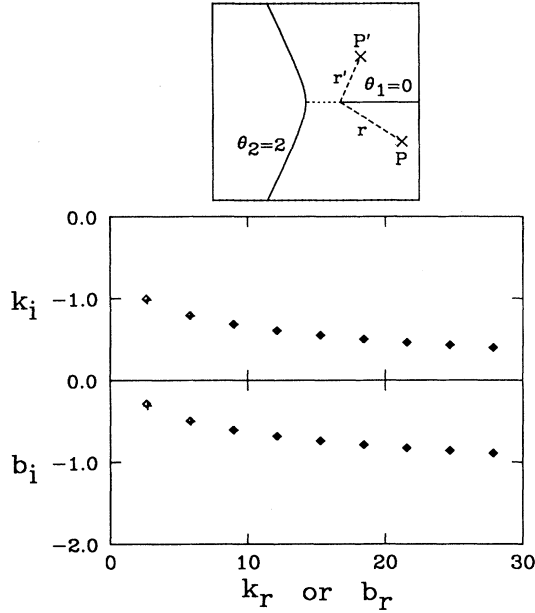


FIG. 2. The same as Fig. 1 for $\theta_1 = 0$ and $\theta_2 = 2.0$. The dotted line in the top box is the diffractive periodic orbit. The crooked dashed line is a typical diffractive path connecting the points P and P' . Its Green function is given by Eq. (10).

hyperbolas is 1. It follows that $a = 1/(\cos \theta_1 - \cos \theta_2)$.

Using the separation $\Psi(\vec{r}) = M(\mu)\Theta(\theta)$ and the expression $\nabla^2 = \left(\frac{\partial^2}{\partial \mu^2} + \frac{\partial^2}{\partial \theta^2}\right)/a^2(\cosh^2 \mu - \cos^2 \theta)$, we obtain the Mathieu equations

$$\frac{d^2 \Theta}{d\theta^2} + a^2(b^2 - k^2 \cos^2 \theta)\Theta = 0, \quad (1a)$$

$$\frac{d^2 M}{d\mu^2} + a^2(-b^2 + k^2 \cosh^2 \mu)M = 0, \quad (1b)$$

where $a^2 b^2$ is a separation constant. Although we can work with these equations, it is convenient to express Eq. (1b) as a one-dimensional potential problem. This is done by a change of coordinates $\zeta = a \sinh \mu$ and $M(\mu) = Z(\zeta)/(\zeta^2 + a^2)^{1/4}$ yielding

$$\frac{d^2 Z}{d\zeta^2} + [k^2 - 2V(\zeta)]Z = 0, \quad (2)$$

with an effective potential

$$V(\zeta) = \frac{1}{2} \left(\frac{b^2 a^2 + 1/2}{\zeta^2 + a^2} - \frac{3\zeta^2}{4(\zeta^2 + a^2)^2} \right). \quad (3)$$

The asymptotic boundary condition is $Z(\zeta) \sim \exp(ik\zeta)$. There is a $y \rightarrow -y$ symmetry so there are two symmetry classes; $Z'(0) = 0$ for even states and $Z(0) = 0$ for odd states. The effective potential is repulsive so there are no bound states, only resonances.

Resonances occur for negative imaginary k which means that asymptotically the wave function is increasing exponentially [7] as $O(\exp(k_i \zeta))$, where $k = k_r - ik_i$. Such wave functions are called Siegert or Gamow states and their asymptotic nature is a reflection of the time-

dependent decay. To work with such wave functions numerically, one must complexify the coordinate $\zeta = |\zeta| \exp(i\alpha)$ [8]. Asymptotically the wave functions are decaying exponentially as a function of $|\zeta|$ if $\alpha > -\arg(k)$.

We numerically solve the simultaneous equations for k and b by shooting [9]. The exact spectra for two choices of parameters are shown in Figs. 1 and 2 as open diamonds. Selected data are also listed in Tables I and II. In Fig. 1, $\theta_1 = 0.4$ and $\theta_2 = 0.9$ which is a smooth system since the radii of curvature of the hyperbolas are comparable to the interhyperbola spacing of 1. In Fig. 2, $\theta_1 = 0$ and $\theta_2 = 2.0$ which is sharp since the right hyperbola is a half plane and its radius of curvature is zero. As will be discussed, the theoretical understanding of the two cases is quite different. For each value of k there is one corresponding value of b . The spectra of k and b are similar in both cases; their real components are very close and their imaginary components differ approximately by a constant.

Periodic orbit theory works by approximating the trace of the Green function, the poles of which are bound states or scattering resonances [1]. This has been worked out for the two disk problem [4] in terms of the single periodic orbit in the system. The result, adapted to this system and including both symmetry classes, is

$$k_{nj} = (n+1)\pi - i \frac{2j+1}{4} \ln \Lambda + \dots, \quad (4)$$

where $n, j = 0, 1, \dots$ and $\Lambda = \tan^2(\theta_2/2)/\tan^2(\theta_1/2)$ is the stability for a complete traversal [10]. The $y \rightarrow -y$ parity is $(-1)^j$.

Equations (1a) and (2) can also be solved using WKB theory. The details will be presented elsewhere [10]. We begin with the expansion

$$\begin{aligned} k &= k_r - ik_i + O(k^{-1}), \\ b &= b_r - ib_i + O(b^{-1}). \end{aligned} \quad (5)$$

b is the eigenvalue of the equation $(\mathcal{L} + b^2)\Psi = 0$ where $\mathcal{L} = (\cosh^2 \mu \frac{\partial^2}{\partial \theta^2} + \cos^2 \theta \frac{\partial^2}{\partial \mu^2})/(\cosh^2 \mu - \cos^2 \theta)$ as can be seen from Eq. (1). In the neighborhood of the periodic orbit and to leading order, the equation for b is the same as the equation for k so that $b_r = k_r$.

The WKB approximation for (1a) is [11]

$$(n+1)\pi = a \int_{\theta_1}^{\theta_2} d\theta \sqrt{b^2 - k^2 \cos^2 \theta}. \quad (6)$$

TABLE I. The exact and semiclassical predictions of k for a selection of resonances and for the geometry of Fig. 1.

n	j	k	k^{sc}
0	0	3.0587 -i0.3930	3.1416 -i0.4342
2	0	9.3881 -i0.4258	9.4248 -i0.4342
10	0	34.5467 -i0.4334	34.5575 -i0.4342
20	0	65.9677 -i0.4340	65.9734 -i0.4342
6	0	21.9714 -i0.4326	21.9911 -i0.4342
6	6	21.2799 -i5.5685	21.9911 -i5.6443
6	12	19.4438 -i10.4733	21.9911 -i10.8544

TABLE II. As in Table I for the geometry of Fig. 2.

n	j	k	k^{sc}
0	0	2.6537 -i1.0060	2.7489 -i1.0201
2	0	8.9913 -i1.3160	9.0321 -i1.3175
10	0	34.1514 -i1.6501	34.1648 -i1.6501
20	0	65.5731 -i1.8131	65.5807 -i1.8131

This equation can be expressed in terms of incomplete elliptic integrals [12]. We expand the integrand of Eq. (6) in powers of $(b^2 - k^2)/(k^2 \sin^2 \theta)$ and keep the first two terms. This gives $k_r = (n + 1)\pi$ and $b_i = f k_i$, where $f = (1 - 2/a \ln \Lambda)$. The imaginary components of k and b are related by a factor f which depends only on the geometry and is the same for all resonances.

The WKB approximation for the resonances of Eq. (2) is [11]

$$(2j + 1)\pi = 2a \int_{z_-}^{z_+} dz \sqrt{k^2 - 2V(z)}, \quad (7)$$

where $V(z)$ is shown in Eq. (3). This formula comes from identifying the resonances as the poles of the transmission coefficient. z_{\pm} are complex turning points which are solutions of $V(z_{\pm}) = k^2/2$. We arrive at a solution in which k is as in Eq. (4) and

$$b_{nj} = (n + 1)\pi - if \frac{2j + 1}{4} \ln \Lambda + \dots \quad (8)$$

These results are consistent with periodic orbit theory. However the calculation of b is new. The imaginary component of b is less negative than that of k while the real components are equal to leading order. The semiclassical results are shown in Fig. 1 as crosses and are also listed in Table I. The worsening of the agreement as we go down in the k plane is expected [13] and is consistent with the expansion (5).

It is interesting to study the higher order terms in the expansion of the resonances. The first two corrections to the leading family of resonances yield [14, 15]

$$k_n = (n + 1)\pi - \frac{c_r}{(n + 1)\pi} - i \left(\frac{\ln \Lambda}{4} - \frac{c_i}{(n + 1)^2 \pi^2} \right), \quad (9)$$

where c_r and c_i are constants. For the symmetric case $\theta_2 = \pi - \theta_1$, we have [14] $c_r = \cot^2 \theta_1$. For the example $\theta_1 = 1.0$ we fit the 200th resonance to Eq. (9) to extract $c_r = 0.4123$ in agreement with the expected result. We also find $c_i = 0.593$ but there is as yet no analytical result. The two constants have been worked out for the two disk problem [15] where it is found that c_i is positive. This means that the resonance lifetimes are longer than the first order approximation, as is also found here. Since periodic orbit theory is an asymptotic expansion in \hbar [16] we expect that the higher order corrections will eventually blow up for any resonance n . The results for the scaling with j are ambiguous so we leave this for a later publication.

Note that some of the resonances are not in the phys-

ical region $k_r > 0$ but instead have $k_r < 0$. (Because the dimension is even, the pole structure is not symmetric with respect to reflections through the imaginary k axis [17].) For example, the state with indices $n = 0$ and $j = 4$ does not exist as a physical resonance although there is a semiclassical prediction. More resonances lie in the unphysical region for large j and for small θ_1 .

For the configuration of Fig. 2 we need a different analysis since Eq. (4) predicts infinite widths. The previous results relied on an expansion in powers of $1/\sin^2 \theta$ which breaks down if $\theta_1 \ll 1$ or $\pi - \theta_2 \ll 1$. In that case we can approximate the hyperbola by a wedge so that a qualitatively different mechanism is responsible for the resonances. There is a class of orbits which are not classical trajectories but rather are diffractive [18] as shown in Fig. 2. The ray starting at P illuminates the vertex which then acts as a point source for an outgoing circular wave which illuminates P' . The Green function for this process [5, 18] is approximately

$$G(P', P, k) = dG(P', V, k)G(V, P, k), \quad (10)$$

where $G(V, P, k)$ is the free space Green function which connects point P to vertex V and can be approximated by the standard Van Vleck formula. The geometric factor d depends on the incoming and outgoing angles and on the wedge angle.

As with geometric orbits [1] taking the trace of G selects the closed orbits. These are all repetitions of the primitive orbit which diffracts from the wedge and bounces back off the wall, with a sign change. When the left wall is straight the primitive orbit contributes to the trace an amount

$$g_0(k) = \frac{d}{\sqrt{16\pi k^3}} \exp(2ik - i\pi/4). \quad (11)$$

For normal incidence $d = 4 \sin(\pi/\gamma)/\gamma$ [10] with $\gamma = 2(1 - \theta_1/\pi)$. Note that $d = 2$ for $\theta_1 = 0$. We must also include the focusing properties of the left wall. The theory of Ref. [18] relies on a picture of cones such that amplitudes decrease with the square root of the cone widths. For a straight wall, a cone leaving the vertex with an opening angle ϕ returns with a width 2ϕ . For a curved wall, the cone returns with a width $2\phi(1 + 1/R)$ where $R = -\sin \theta_2 \tan \theta_2 / (1 - \cos \theta_2)$ is the radius of curvature of the left wall. Therefore, the amplitude is divided by a factor of $\sqrt{1 + 1/R}$ and the denominator of Eq. (11) is replaced by $\sqrt{16\pi k^3(1 + 1/R)}$. This factor can also be derived from the integration perpendicular to the diffractive orbit when the trace is evaluated but the geometric argument above is clearer.

Each multiple of the primitive orbit contributes to the trace an amount equal to g_0 times a power of

$$t = \frac{d}{\sqrt{16\pi k(1 + 1/R)}} \exp(2ik + i\pi/4) \quad (12)$$

which is the approximate Green function to diffract from the vertex to the vertex via the wall (see also Ref. [5]). Therefore we find

$$\text{tr}G(k) = g_0(k) \sum_{r=0}^{\infty} t^r = \frac{g_0(k)}{1 - t}. \quad (13)$$

The poles are given by $t = 1$ whose solution to leading order is

$$k_n = \left(n + \frac{7}{8}\right) \pi - \frac{i}{4} \ln \frac{16\pi^2(n + 7/8)(1 + 1/R)}{d^2}. \quad (14)$$

The factor of $7/8$ comes from the fact that there are two direct Green functions in Eq. (10) and is an important qualitative distinction from the geometric theory. A simple extension to the case where both hyperbolas are sharp yields

$$k_n = \left(n + \frac{3}{4}\right) \pi - \frac{i}{2} \ln \frac{8\pi^2(n + 3/4)}{d_1 d_2}. \quad (15)$$

A WKB analysis is possible for the sharp geometry. For simplicity, we just discuss the result for $\theta_1 = 0$. Then the integral in Eq. (6) can be expressed as a complete elliptic integral of the second kind [12] and expanded in powers and logarithms of $(b^2 - k^2)/b^2$. Again we use the expansion (5) and argue that $b_r = k_r$. The WKB analysis of Eq. (7) is the same as before. The result of these calculations is that k is given as in Eq. (14) and $b_n = k_n + i/2a$. As with the geometric case, the imaginary component of b is less negative than the imaginary component of k .

The results of the semiclassical analysis for the diffractive case are shown in Fig. 2 and in Table II. We find good agreement with the exact results for both k and b .

As in the geometric case, the exact lifetimes are longer than the lowest order semiclassical prediction. The theory predicts only one family of resonances. This is consistent with the numerics where we see that the widths of the next family increase logarithmically as $\theta_1 \rightarrow 0$.

There are two unexplained features of the spectrum for θ_1 small but nonzero. One is the crossover to the geometric result near $|k| \sim 1/(\pi\theta_1^2)$ (by the criterion that $4\pi|k| \sim \Lambda$). The other is the positions of the lower families. These might be explained by including both geometric and diffractive orbits in the Green function. These calculations should be possible for the WKB analysis as well. Another interesting issue is the higher order corrections which are calculable using either periodic orbit theory or WKB. We can also use periodic orbit theory to approximate the resonance spectrum in a system comprised of wedges of finite angles. These issues will be addressed in future presentations.

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